A Gradient Descent Approach for Multi-Objective Picture-Fuzzy Stochastic Programming Problems

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Abstract. This paper explores multi-objective stochastic fuzzy linear programming problems using picture-fuzzy theory to model parameter uncertainty and fuzziness. Picture-fuzzy theory provides a flexible way to handle uncertain data by representing acceptance, rejection, and hesitation degrees. Our study introduces a method to transform the initial stochastic fuzzy problem into a quasiconvex programming problem. Our study enhances computational efficiency and ensures algorithm convergence by applying a gradient descent approach rather than conventional heuristic methods. Our study provides theoretical proof using quasiconvex optimization to validate the proposed method, establishing a foundation for its convergence and effectiveness. To illustrate the method's feasibility and efficacy, the paper presents computational examples demonstrating its correctness and potential applications, particularly in economics and finance where uncertainty and fuzziness in market data are significant. The research opens new pathways for solving complex programming problems in uncertain environments.

Keywords: Multiobjective stochastic linear programming · Picture Fuzzy decision · Probability maximization · Gradient Descent · Variance covariance matrices.

1 Introduction

This paper addresses the complex challenge of multi-objective stochastic linear programming, where the problem becomes more complicated due to the randomness not only of vector b in the inequality $A \leq b$, but also of matrix A, whereas previous studies have mentioned this but focused only on solving with the stochastic component b (see [4, 5]). Such randomness in both components significantly increases the complexity of the constraint space. Despite these complexities, we have successfully demonstrated that the overall feasible set remains convex, an essential property for applying advanced optimization techniques.

To effectively tackle this increased complexity, we propose a new approach using picture fuzzy theory (see [6]). This theory extends traditional fuzzy logic by incorporating degrees of acceptance, rejection, and hesitation, thus providing a richer framework for representing uncertainty. The improvements include the use of monotonic membership functions in picture fuzzy sets, ensuring that as the uncertainty in the parameters decreases, the membership values adjust in a predictable and consistent manner. This feature is crucial for maintaining the integrity of decision-making processes under uncertainty, especially in multi-objective situations. This is the key difference between our research and previous studies that have employed non-monotonic functions such as Gaussian (see [18]), trapezoidal (see [16, 17]), etc.

Our paper further proposes a methodological transformation of the original stochastic fuzzy problem into a quasiconvex programming problem. To solve this type of problem, previous studies have used traditional heuristic methods (see [13, 14]). Nonetheless, we have shown that this class of issues can be resolved using a gradient descent approach (see [12, 19]), which is a robust and widely applied method in optimization problems. To ensure that the proposed gradient method can solve the proposed problem, we provide rigorous theoretical proof using quasiconvex optimization techniques. This proof establishes a solid foundation for the convergence and effectiveness of the algorithm, ensuring that the derived solutions are both theoretically robust and operationally viable. To demonstrate the practicality and accuracy of our approach, the paper includes computational examples that illustrate the validity of the methodology.

This problem is particularly relevant to the fields of economics and finance, where portfolio optimization and other financial instruments often grapple with significant uncertainty and ambiguity in market data. The objective function in this problem is the Sharpe Ratio (SR) (see [15]), which is a widely recognized measure for evaluating risk-adjusted returns, quantifying the ratio of expected returns to the portfolio's standard deviation. While SR is extensively used in financial applications and carries substantial economic relevance, its non-convex nature poses challenges when directly used in optimization problems. Nevertheless, this paper demonstrates that objective functions like SR can still be efficiently and globally solved using existing convex programming techniques (see also [12]). The global convergence of the solution is guaranteed.

Regarding the structure of this paper, we have organized it into key sections to facilitate a thorough understanding of the concepts and methodologies employed. In Section 2, we delve into the foundational theories of fuzzy logic and quasiconvex optimization, setting the stage for their application in complex stochastic problems. Section 3 is dedicated to presenting the general framework of the multi-objective stochastic linear optimization problem, detailing how we incorporate picture-fuzzy sets to model the inherent uncertainty. This section also includes the fuzzification process of both the objective functions and constraints to adequately capture the nuances of randomness in A and b. To demonstrate the practical application and effectiveness of our proposed method, we conclude Section 3 with computational example that highlight how our approach can be effectively applied.

2 Preliminaries: Concepts and definitions

2.1 Picture Fuzzy

Given a universal set $\mathbb{U}$, a generalization of fuzzy sets that introduces more granularity and flexibility in representing uncertainty is called a picture fuzzy set of $\mathbb{U}$, denoted by $\tilde{S}$. This set is described by three distinct mappings: the positive membership function $\mu_{\tilde{S}} : \mathbb{U} \to [0,1]$, the neutral membership function $\eta_{\tilde{S}} : \mathbb{U} \to [0,1]$, and the negative membership function $\nu_{\tilde{S}} : \mathbb{U} \to [0,1]$. The definition of the picture fuzzy set $\tilde{S}$ is formalized as follows:

Definition 1 (see [6]). *Given the condition $0 \le \mu_{\tilde{S}}(\theta) + \eta_{\tilde{S}}(\theta) + \nu_{\tilde{S}}(\theta) \le 1$ for all $\theta \in \mathbb{U}$, the picture fuzzy set $\tilde{S}$ is given by:*

$$\tilde{S} = \{(\theta, \mu_{\tilde{S}}(\theta), \eta_{\tilde{S}}(\theta), \nu_{\tilde{S}}(\theta)) \mid \theta \in \mathbb{U}\} \tag{1}$$

We can now express the general optimization problem with the following formulation:

$$\begin{aligned} \min \quad & F(\boldsymbol{\theta}) \\ \text{s. t} \quad & z_t(\boldsymbol{\theta}) \le 0, \quad t = 1, \dots, k, \\ & \boldsymbol{\theta} \in \mathcal{X}, \end{aligned} \tag{2}$$

where $\mathcal{X}$ denotes a compact convex set ($\mathcal{X} \neq \emptyset$).

This study employs a fuzzy optimization model that integrates fuzzy components, treating the variables and parameters as exact values. This is termed flexible optimization (see [1]). Two categories of fuzzy components exist. The first concerns the objective function (indicated by ($\widetilde{\min}$ or $\widetilde{\max}$)), while the second applies to a fuzzy relation (such as ($\preceq, \simeq, \succeq$)). A fuzzy optimization problem comprising fuzzy elements can be expressed as follows:

$$\begin{aligned} \widetilde{\min} \quad & F(\boldsymbol{\theta}) \\ \text{s. t} \quad & z_t(\boldsymbol{\theta}) \preceq 0, \quad t = 1, \dots, k, \\ & \boldsymbol{\theta} \in \mathcal{X} \end{aligned} \tag{3}$$

In this formulation, the symbols "$\widetilde{\min}$" and "$\preceq$" refer to the fuzzy versions of "minimize" and "less than or equal to," respectively, implying that the objective function should be minimized as much as feasible and that the constraints should be as well accepted as possible. Fuzzy optimization approaches with fuzzy objectives are widely utilized in decision-making for real-world applications.

2.2 Quasiconvexity

Definition 2. ***(Pseudoconvex function (see [7])).*** *Let $\mathcal{X}$ denote a nonempty convex set, and let $f : \mathbb{R}^n \to \mathbb{R}$ represent a differentiable function defined* *on $\mathcal{X}$. We characterize f as a pseudoconvex function on $\mathcal{X}$ If for each $\theta^1, \theta^2 \in \mathcal{X}$, the subsequent condition is satisfied:*

$$f(\theta^2) < f(\theta^1) \Rightarrow \langle \nabla f(\theta^1), \theta^2 - \theta^1 \rangle < 0. \tag{4}$$

If f is pseudoconvex, then the negative of f, i.e., $-f$, is referred to as a pseudoconcave function.

Definition 3 (Semistrictly quasiconvex function (see [2])). *Let $\mathcal{X} \subset \mathbb{R}^n$ be a convex set, and f be defined on $\mathcal{X}$. If for any $\theta^1, \theta^2 \in \mathcal{X}$ and $0 < \lambda < 1$, the condition*

$$f(\theta^1) > f(\theta^2) \tag{5}$$

implies that

$$f(\lambda\theta^1 + (1-\lambda)\theta^2) < f(\theta^1), \tag{6}$$

then f is called semistrictly quasiconvex.

Definition 4 (Semistrictly quasiconvex programming problem). *The semistrictly quasiconvex programming problem is articulated as follows:*

$$\min f(\theta) \quad s.t. \quad \theta \in \mathcal{X}, \tag{SQP}$$

where $\mathcal{X} \subset \mathbb{R}^n$ is a convex set, and $f(\theta)$ is semistrictly quasiconvex on $\mathcal{X}$.

Proposition 1. *In the semistrictly quasiconvex programming problem* (SQP)*, any local minimum is also a global minimum (see [1]). In light of this, convex programming methods are a powerful tool for solving this challenge.*

3 Methodology and problem formulation

3.1 Multiobjective Stochastic Linear Programming Problems

We focus on multiobjective stochastic linear programming in this part. The issue at hand can be stated as:

$$\max \overline{\boldsymbol{C}}\boldsymbol{\theta} = (\overline{\boldsymbol{c}}_1\boldsymbol{\theta}, \ldots, \overline{\boldsymbol{c}}_k\boldsymbol{\theta}) \quad \text{s.t.} \quad \overline{\boldsymbol{A}}\boldsymbol{\theta} \leq \overline{\boldsymbol{b}}, \quad \boldsymbol{\theta} \geq 0, \tag{\textbf{MSLP}}$$

where $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)^T$ is an n-dimensional decision variable vector, $\overline{\boldsymbol{c}}_i = (\bar{c}_{i1}, \ldots, \bar{c}_{in})$ is an n-dimensional random variable vector, $\bar{c}_{i\ell}, \ell = 1, \ldots, n, i = 1, \ldots, k$ are Gaussian random variables, represented as $\mathcal{N}(m_{\bar{c}_{i\ell}}, \sigma_{i\ell\ell})$. The variance-covariance matrix $\boldsymbol{V}_{\bar{\boldsymbol{c}}_i}, i = 1, \ldots, k$, describing the relationship between the Gaussian random variables, is defined as:

$$\boldsymbol{V}_{\bar{\boldsymbol{c}}_i} = [\sigma_{izt}]_{n \times n}, i = 1, \ldots, k. \tag{7}$$

The predicted value of the row vector $\overline{\boldsymbol{c}}_i$ with respect to the random variable is represented by:

$$\boldsymbol{m}_{\overline{\boldsymbol{c}}_i} = (m_{\bar{c}_{i1}}, \cdots, m_{\bar{c}_{in}}), \tag{8}$$

and with the covariance matrix $\boldsymbol{V}_{\overline{\boldsymbol{c}}_i}$, the random variable vector $\overline{\boldsymbol{c}}$ can be expressed as:

$$\overline{\boldsymbol{c}}_i \sim \mathcal{N}\left(\boldsymbol{m}_{\overline{\boldsymbol{c}}_i}, \boldsymbol{V}_{\overline{\boldsymbol{c}}_i}\right). \tag{9}$$

The objective function $\overline{\boldsymbol{c}}_i\boldsymbol{\theta}$ also follows a Gaussian distribution, based on the features of Gaussian random variables in general:

$$\overline{\boldsymbol{c}}_i\boldsymbol{\theta} \sim \mathcal{N}\left(\boldsymbol{m}_{\overline{\boldsymbol{c}}_i}\boldsymbol{\theta}, \boldsymbol{\theta}^T\boldsymbol{V}_{\overline{\boldsymbol{c}}_i}\boldsymbol{\theta}\right), \tag{10}$$

Conventional mathematical programming techniques are inapplicable to the multiobjective stochastic linea r programming (**MSLP**) issue because it incorporates random variables into the objective functions and restrictions. We deal with the limitations of (**MSLP**) as conditions that are confined by chance (see [5]) in order to solve this problem. A certain probability β_j or above, referred to as the constraint probability level, is required for the j-th constraint of (**MSLP**) to hold. The definition of the set $\mathcal{X}(\boldsymbol{\beta})$ that meets this requirement is:

$$\mathcal{X}(\boldsymbol{\beta}) = \left\{\boldsymbol{\theta} \in \mathbb{R}^n, \boldsymbol{\theta} \geq 0 \mid P\left(\overline{\boldsymbol{a}}_j\boldsymbol{\theta} \leq \bar{b}_j\right) \geq \beta_j, j = 1, \cdots, m\right\} \tag{11}$$

where $\overline{\boldsymbol{a}}_j = (\bar{a}_{j1}, \ldots, \bar{a}_{jn})$ is the j-th row vector of random variables from matrix $\overline{\boldsymbol{A}}$, and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_m)$.

To determine the deterministic equivalent constraints for the chance - constrained formulation, we first assume that only $\bar{b}_i$ is a random variable on the right-hand side, and that $\bar{a}_{ij} = a_{ij}$ is constant. Thus, we use the notation $\bar{a}_{ij} = a_{ij}$. Let $F_i(\tau)$ represent the cumulative distribution function of $\bar{b}_i$. Since

$$P\left(\sum_{j=1}^{n} a_{ij}\theta_j \leq \bar{b}_i\right) = 1 - F_i\left(\sum_{j=1}^{n} a_{ij}\theta_j\right), i = 1, \ldots, m \tag{12}$$

the chance constraints (11) can be rewritten as:

$$F_i\left(\sum_{j=1}^{n} a_{ij}\theta_j\right) \leq 1 - \beta_i, i = 1, \ldots, m \tag{13}$$

Let $K_{1-\beta_i}$ denote the maximum value of τ such that $\tau = F_i^{-1}(1-\beta_i)$. Therefore, the inequality in (13) can be expressed as:

$$\sum_{j=1}^{n} a_{ij}\theta_j \leq K_{1-\beta_i}, i = 1, \ldots, m \tag{14}$$

We may describe the probability as follows if we assume that $\bar{b}_i$ is a normally distributed random variable with a mean of $m_{\bar{b}_i}$ and variance $\sigma^2_{\bar{b}_i}$:

$$\begin{aligned} P\left(\sum_{j=1}^{n} a_{ij}\theta_j \leq \bar{b}_i\right) &= P\left(\frac{\bar{b}_i - m_{\bar{b}_i}}{\sigma_{\bar{b}_i}} \geq \frac{\sum_{j=1}^{n} a_{ij}\theta_j - m_{\bar{b}_i}}{\sigma_{\bar{b}_i}}\right) \\ &= 1 - \Phi\left(\frac{\sum_{j=1}^{n} a_{ij}\theta_j - m_{\bar{b}_i}}{\sigma_{\bar{b}_i}}\right), \end{aligned} \tag{15}$$

where Φ represents the cumulative distribution function of the standard normal distribution $\mathcal{N}(0,1)$. Thus, the chance constraint (15) can be reformulated as:

$$\sum_{j=1}^{n} a_{ij}\theta_j \leq m_{\bar{b}_i} + \sigma_{\bar{b}_i}\Phi^{-1}\left(1-\beta_i\right), i=1,\ldots,m \tag{16}$$

where Φ^{-1} denotes the inverse cumulative distribution function of the standard normal distribution.

Now, in the scenario where both $\bar{b}_i$ and $\bar{a}_{ij}$ are normally distributed, we define $m_{\bar{b}_i}$ and $\sigma^2_{\bar{b}_i}$ as the mean and variance of $\bar{b}_i$, respectively, and $m_{\bar{a}_{ij}}$ and $V_{\bar{\boldsymbol{a}}_i}$ as the mean and variance-covariance matrix of $\bar{a}_{ij}$. Assume that $\bar{b}_i$ and $\bar{a}_{ij}$ are independent; the random variable

$$\frac{\bar{b}_i - \bar{\boldsymbol{a}}_i\boldsymbol{\theta} - \left(m_{\bar{b}_i} - \boldsymbol{m}_{\bar{\boldsymbol{a}}_i}\boldsymbol{\theta}\right)}{\sqrt{\sigma^2_{\bar{b}_i} + \boldsymbol{\theta}^T V_{\bar{\boldsymbol{a}}_i}\boldsymbol{\theta}}}, i=1,\ldots,m \tag{17}$$

adheres to a conventional Gaussian distribution Standard normal distribution with a mean of 0 and a variance of 1. Therefore, the probability $P(\bar{\boldsymbol{a}}_i\ \boldsymbol{\theta} \leq \bar{b}_i)$ can be reformulated as:

$$\begin{aligned} &P\left(\frac{\bar{b}_i - \sum_{j=1}^{n}\bar{a}_{ij}\theta_j - \left(m_{\bar{b}_i} - \sum_{j=1}^{n} m_{\bar{a}_{ij}}\theta_j\right)}{\sqrt{\sigma^2_{\bar{b}_i} + \boldsymbol{\theta}^T V_{\bar{\boldsymbol{a}}_i}\boldsymbol{\theta}}} \geq \frac{-\left(m_{\bar{b}_i} - \sum_{j=1}^{n} m_{\bar{a}_{ij}}\theta_j\right)}{\sqrt{\sigma^2_{\bar{b}_i} + \boldsymbol{\theta}^T V_{\bar{\boldsymbol{a}}_i}\boldsymbol{\theta}}}\right) \\ &= 1 - \Phi\left(\frac{\sum_{j=1}^{n} m_{\bar{a}_{ij}}\theta_j - m_{\bar{b}_i}}{\sqrt{\sigma^2_{\bar{b}_i} + \boldsymbol{\theta}^T V_{\bar{\boldsymbol{a}}_i}\boldsymbol{\theta}}}\right), \end{aligned} \tag{18}$$

which leads to the following transformed inequality for the chance constraint:

$$\sum_{j=1}^{n} m_{\bar{a}_{ij}}\theta_j - \Phi^{-1}\left(1-\beta_i\right)\sqrt{\sigma^2_{\bar{b}_i} + \boldsymbol{\theta}^T V_{\bar{\boldsymbol{a}}_i}\boldsymbol{\theta}} \leq m_{\bar{b}_i}, i=1,\ldots,m. \tag{19}$$

Proposition 2. *With constraint probability level* $\boldsymbol{\beta} = (\beta_1,\ldots,\beta_m)$ *and* $0 \leq \beta_i \leq 0.5$, $\sum_{j=1}^{n} m_{\bar{a}_{ij}}\theta_j - \Phi^{-1}\left(1-\beta_i\right)\sqrt{\sigma^2_{\bar{b}_i} + \boldsymbol{\theta}^T V_{\bar{\boldsymbol{a}}_i}\boldsymbol{\theta}}$ *is convex and the constraint set* $\mathcal{X}(\boldsymbol{\beta})$ *is a convex set.*

According to the previously stated chance-constrained conditions (11), utilizing a probability maximization strategy for the objective function in (**MSLP**) allows us to reformulate the maximizing of the objective functions $\overline{\boldsymbol{C}}\boldsymbol{\theta}$, into maximizing the probability that each objective function $\bar{\boldsymbol{c}}_i\boldsymbol{\theta}$ meets or exceeds a certain acceptable threshold $\hat{f}_i$, which is referred to as the permissible objective level. This probability function is expressed as:

$$\kappa_i\left(\boldsymbol{\theta}, \hat{f}_i\right) \stackrel{\text{def}}{=} P\left(\bar{\boldsymbol{c}}_i\boldsymbol{\theta} \geq \hat{f}_i\right). \tag{20}$$

Consequently, the problem (**MSLP**) can be restructured as:

$$\max_{\boldsymbol{\theta}\in\mathcal{X}(\boldsymbol{\beta})} (\kappa_1(\boldsymbol{\theta},\hat{f}_1),\ldots,\kappa_k(\boldsymbol{\theta},\hat{f}_k)) \qquad (\textbf{MOP-P}(\hat{\boldsymbol{f}},\beta))$$

From (10), the objective function $\kappa_i\left(\boldsymbol{\theta},\hat{f}\right)$ can be rewritten as:

$$\begin{aligned}\kappa_i\left(\boldsymbol{\theta},\hat{f}_i\right) &= P\left(\frac{\bar{\boldsymbol{c}}_i\boldsymbol{\theta} - E\left[\bar{\boldsymbol{c}}_i\right]\boldsymbol{\theta}}{\sqrt{\boldsymbol{\theta}^T\boldsymbol{V}_{\bar{\boldsymbol{c}}_i}\boldsymbol{\theta}}} \geq \frac{\hat{f}_i - E\left[\bar{\boldsymbol{c}}_i\right]\boldsymbol{\theta}}{\sqrt{\boldsymbol{\theta}^T\boldsymbol{V}_{\bar{\boldsymbol{c}}_i}\boldsymbol{\theta}}}\right)\\ &= 1-\Phi\left(\frac{\hat{f}_i - E\left[\bar{\boldsymbol{c}}_i\right]\boldsymbol{\theta}}{\sqrt{\boldsymbol{\theta}^T\boldsymbol{V}_{\bar{\boldsymbol{c}}_i}\boldsymbol{\theta}}}\right)\end{aligned} \qquad (21)$$

Given that Φ is a monotonically increasing function, the problem $(\textbf{MOP-P}(\hat{\boldsymbol{f}},\beta))$ is equivalent to:

$$\min_{\boldsymbol{\theta}\in\mathcal{X}(\boldsymbol{\beta})} (\mathcal{S}_1(\boldsymbol{\theta}),\ldots,\mathcal{S}_k(\boldsymbol{\theta})) \qquad (\textbf{MOP-P1}(\hat{\boldsymbol{f}},\boldsymbol{\beta}))$$

where $\mathcal{S}_i(\boldsymbol{\theta}) = -\frac{E[\bar{\boldsymbol{c}}_i]\boldsymbol{\theta}-\hat{f}_i}{\sqrt{\boldsymbol{\theta}^T\boldsymbol{V}_{\bar{\boldsymbol{c}}_i}\boldsymbol{\theta}}}, i = 1,\ldots,k$. Here, we observe that the objective function is no longer purely convex but has a more intricate structure, described as quasiconvex.

Proposition 3. *The objective functions* $\mathcal{S}_i(\boldsymbol{\theta}), i = 1, ..., k$ *are pseudoconvex.*

Proof. Let φ_1 and φ_2 be two functions defined on a set X. If φ_1 is positive and concave, and φ_2 is positive and convex on X, with both φ_1 and φ_2 being differentiable on X, then the fractional function φ_1/φ_2 is pseudoconcave on X (refer to [2]). Observe that $E\left[\bar{\boldsymbol{c}}_i\right] - \hat{f}_i$ constitutes a positive linear expression, given that $\hat{f}_i$ is a constant, whereas $\sqrt{\boldsymbol{\theta}^T\boldsymbol{V}_{\bar{\boldsymbol{c}}_i}\boldsymbol{\theta}}$ is convex. Consequently, the function $\dfrac{E\left[\bar{\boldsymbol{c}}_i\right]\boldsymbol{\theta} - \hat{f}_i}{\sqrt{\boldsymbol{\theta}^T\boldsymbol{V}_{\bar{\boldsymbol{c}}_i}\boldsymbol{\theta}}}$ is pseudoconcave, and therefore $\mathcal{S}_i(\boldsymbol{\theta}) = -\dfrac{E\left[\bar{\boldsymbol{c}}_i\right]\boldsymbol{\theta} - \hat{f}_i}{\sqrt{\boldsymbol{\theta}^T\boldsymbol{V}_{\bar{\boldsymbol{c}}_i}\boldsymbol{\theta}}}$ is pseudoconvex. □

3.2 Problem (MSLP) with fuzzy decision making

To address the issue $(\textbf{MOP-P1}(\hat{\boldsymbol{f}},\boldsymbol{\beta}))$, where (**MSLP**) is expressed using a probability maximization model, we present a picture fuzzy decision-making method in this section. The decision-maker needs to know their allowable goal level $\hat{\boldsymbol{f}} = (\hat{f}_1,\ldots,\hat{f}_k)$ in advance in order to solve $(\textbf{MOP-P1}(\hat{\boldsymbol{f}},\boldsymbol{\beta}))$. Both a smaller value of the function $\mathcal{S}_i(\boldsymbol{\theta})$ and a bigger value of the allowed goal level $\hat{f}_i$ are usually preferred by the decision-maker. The objective function $\mathcal{S}_i(\boldsymbol{\theta})$ rises as the allowed objective level $\hat{f}_i$ rises since these preferences are incompatible with one another.

We suggest the following multiobjective programming problem, which is seen as an obvious continuation of (**MOP-P1**$(\hat{\boldsymbol{f}}, \boldsymbol{\beta})$), from this point of view:

$$\min_{\boldsymbol{\theta} \in X(\boldsymbol{\beta}), \hat{f}_i \in \mathbb{R}, i=1,\ldots,k} (\mathcal{S}_1(\boldsymbol{\theta}), \ldots, \mathcal{S}_k(\boldsymbol{\theta}), -\hat{f}_1, \ldots, -\hat{f}_k) \qquad (\textbf{MOP-P2}(\hat{\boldsymbol{f}}, \boldsymbol{\beta}))$$

It is essential to recognize that in (**MOP-P2**$(\hat{\boldsymbol{f}}, \boldsymbol{\beta})$), the allowable objective levels $\hat{f}_i, i = 1, \cdots, k$ are not static values, but are regarded as decision variables to be optimized.

Considering the ambiguity in the decision-maker's evaluation, it is reasonable to deduce that the decision-maker has a picture fuzzy goal for each objective function and a picture fuzzy relation for each constraint in (**MOP-P2**$(\hat{\boldsymbol{f}}, \boldsymbol{\beta})$). Given these ambiguous objectives and relationships, the issue can be reformulated as the subsequent fuzzy multiobjective programming problem:

$$\begin{aligned} &\widetilde{\min}\ (\mathcal{S}_1(\boldsymbol{\theta}), \ldots, \mathcal{S}_k(\boldsymbol{\theta}), \hat{f}_1, \ldots, \hat{f}_k) \\ &\text{s.t. } g_i(\boldsymbol{\theta}) \preceq 0, i = 1, \ldots, m \\ &\quad\ \ \boldsymbol{\theta} \geq 0. \end{aligned} \qquad (\textbf{FMOP1})$$

where $g_i(\boldsymbol{\theta}) = \sum_{j=1}^{n} m_{\bar{a}_{ij}} \theta_j - \Phi^{-1}(1 - \beta_i) \sqrt{\sigma^2_{\bar{b}_i} + \boldsymbol{\theta}^T V_{\bar{\boldsymbol{a}}_i} \boldsymbol{\theta}} - m_{\bar{b}_i}$. Fuzzy decisions are quantified through the use of membership functions. Let us define membership functions for a objective function $\mathcal{S}_i(\boldsymbol{\theta})$ as $(\mu_{\mathcal{S}_i}(\mathcal{S}_i(\boldsymbol{\theta})), \eta_{\mathcal{S}_i}(\mathcal{S}_i(\boldsymbol{\theta})), \nu_{\mathcal{S}_i}(\mathcal{S}_i(\boldsymbol{\theta})))$, membership functions of a permissible objective level $\hat{f}_i$ as $(\mu_{\hat{f}_i}(\hat{f}_i), \eta_{\hat{f}_i}(\hat{f}_i), \nu_{\hat{f}_i}(\hat{f}_i))$, and membership functions of a constraints $g_j(\boldsymbol{\theta})$ as $(\mu_{g_j}(g_j(\boldsymbol{\theta})), \eta_{g_j}(g_j(\boldsymbol{\theta})), \nu_{g_j}(g_j(\boldsymbol{\theta})))$, respectively.

Proposition 4. *Consider a monotonically decreasing function $m(\cdot)$. The positive membership function μ with respect to a function or parameter z is expressed as:*

$$\mu(z) = \begin{cases} 0 & \text{if } z \geq z^0, \\ m(z) & \text{if } z^0 \geq z \geq z^1, \\ 1 & \text{if } z \leq z^1 \end{cases}$$

where z^0 is the maximum value for which $\mu(z) = 0$ and z^1 is the minimum value for which $\mu(z) = 1$. On the other hand, for monotonically increasing functions $n(\cdot)$ and $k(\cdot)$, the neutral and negative membership functions η, ν with respect to z are given as:

$$\eta(z) = \begin{cases} a & \text{if } z \geq z^0, \\ n(z) & \text{if } z^0 \geq z \geq z^1, \\ 0 & \text{if } z \leq z^1 \end{cases}$$

$$\nu(z) = \begin{cases} b & \text{if } z \geq z^0, \\ k(z) & \text{if } z^0 \geq z \geq z^1, \\ 0 & \text{if } z \leq z^1 \end{cases}$$

Here, z^0 represents the minimum value of z when $\eta(z) = a, \nu(z) = b, (a + b \leq 1)$, while z^1 is the maximum value where z if $\eta(z) = 0, \nu(z) = 0$.

The issue (**FMOP1**) can thereafter be reformulated as the following multi-objective programming problem:

$$\begin{aligned}
&\max \left(\mu_{\mathcal{S}_i}\left(\mathcal{S}_i(\boldsymbol{\theta})\right), \mu_{\hat{f}_i}(\hat{f}_i), \mu_{g_j}\left(g_j(\boldsymbol{\theta})\right)\right), i=1,\dots,k, j=1,\dots,m\\
&\min \left(\eta_{\mathcal{S}_i}\left(\mathcal{S}_i(\boldsymbol{\theta})\right), \eta_{\hat{f}_i}(\hat{f}_i), \eta_{g_j}\left(g_j(\boldsymbol{\theta})\right)\right), i=1,\dots,k, j=1,\dots,m\\
&\min \left(\nu_{\mathcal{S}_i}\left(\mathcal{S}_i(\boldsymbol{\theta})\right), \nu_{\hat{f}_i}(\hat{f}_i), \nu_{g_j}\left(g_j(\boldsymbol{\theta})\right)\right), i=1,\dots,k, j=1,\dots,m\\
&\text{s.t. } \boldsymbol{\theta} \geq 0.
\end{aligned} \qquad (\mathbf{FMOP2})$$

Now, by defining $\dot{\mu} = 1 - \mu$, the problem (**FMOP2**) ecomes equivalent to:

$$\begin{aligned}
\min\ &\left(\dot{\mu}_{\mathcal{S}_i}\left(\mathcal{S}_i(\boldsymbol{\theta})\right), \dot{\mu}_{\hat{f}_i}(\hat{f}_i), \dot{\mu}_{g_j}\left(g_j(\boldsymbol{\theta})\right),\right.\\
&\eta_{\mathcal{S}_i}\left(\mathcal{S}_i(\boldsymbol{\theta})\right), \eta_{\hat{f}_i}(\hat{f}_i), \eta_{g_j}\left(g_j(\boldsymbol{\theta})\right),\\
&\left.\nu_{\mathcal{S}_i}\left(\mathcal{S}_i(\boldsymbol{\theta})\right), \nu_{\hat{f}_i}(\hat{f}_i), \nu_{g_j}\left(g_j(\boldsymbol{\theta})\right)\right)\\
&i=1,\dots,k, j=1,\dots,m\\
\text{s.t. }\ &\boldsymbol{\theta} \geq 0.
\end{aligned} \qquad (\mathbf{FMOP3})$$

This multiobjective semistrictly quasiconvex programming issue has been examined in several studies (see [8, 9]). To circumvent the computational complexity of directly addressing this, we propose reformulating problem (**FMOP3**) as follows, inspired by [10]:

$$\begin{aligned}
\min\ &\max\left(\dot{\mu}_{\mathcal{S}_i}\left(\mathcal{S}_i(\boldsymbol{\theta})\right), \dot{\mu}_{\hat{f}_i}(\hat{f}_i), \dot{\mu}_{g_j}\left(g_j(\boldsymbol{\theta})\right),\right.\\
&\eta_{\mathcal{S}_i}\left(\mathcal{S}_i(\boldsymbol{\theta})\right), \eta_{\hat{f}_i}(\hat{f}_i), \eta_{g_j}\left(g_j(\boldsymbol{\theta})\right),\\
&\left.\nu_{\mathcal{S}_i}\left(\mathcal{S}_i(\boldsymbol{\theta})\right), \nu_{\hat{f}_i}(\hat{f}_i), \nu_{g_j}\left(g_j(\boldsymbol{\theta})\right)\right)\\
&i=1,\dots,k, j=1,\dots,m\\
\text{s.t. }\ &\boldsymbol{\theta} \geq 0.
\end{aligned} \qquad (\mathbf{MP})$$

This single-objective problem has garnered significant attention in the literature (see to [11]). The semistrictly quasiconvex characteristics of the objectives introduce complexity to the problem, rendering it tough and inadequately explored in prior research. This paper delineates the characteristics of the issue are as follows.

Proposition 5. *(**MP**) is categorized as a semistrictly quasiconvex programming issue.*

Thang et al. [12] have demonstrated the convergence of the universal solution in the realm of semistrictly quasiconvex programming and devised an algorithm utilizing the gradient direction approach. The efficacy of this approach has been confirmed by computational studies. This study employs the gradient direction method to address (**MP**), following the methodology of [12] and Proposition 5.

3.3 Examples

Example 1. (see [4]) To illustrate the efficacy of our suggested fuzzy decision-making methodology, we analyze the subsequent three-objective stochastic linear

programming problem:
[MOSLP]

$$\max(\overline{\boldsymbol{c}}_1\boldsymbol{\theta}, \overline{\boldsymbol{c}}_2\boldsymbol{\theta}, \overline{\boldsymbol{c}}_3\boldsymbol{\theta}) \quad \text{s.t.} \boldsymbol{a}_j\boldsymbol{\theta} \leq \overline{b}_j, j = 1, 2 \quad \boldsymbol{\theta} \geq 0 \tag{22}$$

where $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3, \theta_4)^T$ is a 4-dimensional decision column vector. The coefficient row vectors $\boldsymbol{a}_j, j = 1, 2$ are defined as $\boldsymbol{a}_1 = (7, 3, 4, 6), \boldsymbol{a}_2 = (-5, -6, -7, -9)$. The Gaussian random variables $(\bar{b}_1, \bar{b}_2)$ with $b_1 \sim \mathrm{N}(27, 6^2), b_2 \sim \mathrm{N}(-15, 7^2)$, $\overline{\boldsymbol{c}}_i, i = 1, 2, 3$ are 4-dimensional Gaussian random variables, with mean vectors $\boldsymbol{E}[\overline{\boldsymbol{c}}_1] = (2, 3, 2, 4), \boldsymbol{E}[\overline{\boldsymbol{c}}_2] = (10, -7, 1, -2)$, $\boldsymbol{E}[\overline{\boldsymbol{c}}_3] = (-8, -5, -7, -14)$. The variance-covariance matrices $V_i, i = 1, 2, 3$, are as follows:

$$V_1 = \begin{pmatrix} 25 & -1 & 0.8 & -2 \\ -1 & 4 & -1.2 & 1.2 \\ 0.8 & -1.2 & 4 & 2 \\ -2 & 1.2 & 2 & 9 \end{pmatrix}, \qquad V_2 = \begin{pmatrix} 16 & 1.4 & -1.2 & 1.4 \\ 1.4 & 1 & 1.5 & -0.8 \\ -1.2 & 1.5 & 25 & -0.6 \\ 1.4 & -0.8 & -0.6 & 4 \end{pmatrix}$$

$$V_3 = \begin{pmatrix} 4 & -1.9 & 1.5 & 1.8 \\ -1.9 & 25 & 0.8 & -0.4 \\ 1.5 & 0.8 & 9 & 2.5 \\ 1.8 & -0.4 & 2.5 & 36 \end{pmatrix}$$

For this issue, let's pretend the decision maker has $\boldsymbol{\beta} = (\beta_1, \beta_2) = (0.7, 0.7)$ as the levels of the constraint probability.

To validate the efficiency of our proposed method, we compare it with previous methods (see [4]), including fuzzy decision making and the probability maximization model.

	Proposed Methods	Yano's Method	Probability max.
λ^*	0.546523	0.536605	0.173128
f_1^*	30.1215	28.0819	25
f_2^*	-30.3785	-29.3565	-35
f_3^*	-31.6524	-32.7991	-35
$\mu_{f_1}(f_1^*)$	0.543432	0.536605	0.596050
$\mu_{f_2}(f_2^*)$	0.542875	0.536605	0.607070
$\mu_{f_3}(f_3^*)$	0.537543	0.536605	0.570710

The results from the proposed method demonstrate several advantages compared to the two previous methods (Yano's method and the probability maximization method). First, the λ^* value in the proposed method is 0.546523, higher than Yano's method (0.536605) and the probability maximization method (0.173128), indicating a better optimal performance. These results were achieved by employing a fuzzy relation method to extend the acceptable set, which led to a better optimization outcome. This approach allowed for more flexibility and precision in handling uncertainties, ultimately improving the overall performance.

4 Conclusion

This study introduces a multi-objective stochastic linear optimization model designed to maximize probability and threshold values, thus facilitating decision-making through picture fuzzy relations. We have established that the deterministic problem constitutes a semistrictly quasiconvex programming issue and have introduced a method utilizing strictly monotonic membership functions to convert it into a flexible variant, which is also demonstrated to be semistrictly quasiconvex. The picture fuzzy multi-objective stochastic optimization issue can be effectively resolved utilizing gradient descent techniques, which demand considerably less resources than the previously utilized genetic algorithms. This method possesses significant practical benefits and the prevalent presence of semistrictly quasiconvex functions in real-world applications, indicating its extensive applicability in diverse mathematical models incorporating semistrictly quasiconvex functions.

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